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INEQUALITIES FOR COORDINATED HARMONIC PREINVEX FUNCTIONS

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Abstract. In this paper, we introduce and investigate the co-ordinated harmonic preinvex functions. Some new Hermite-Hadamard inequalities for co-ordinated harmonic preinvex functions are derived. These new results can be viewed as significant refinement and improvement of the known results. Some special cases are discussed as applications of main results. The ideas and techniques of this paper may stimulate further research in this area.

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1. Introduction

Recently, much attention have been given to derive several integral type inequalities related to various type of convex functions using new ideas and innovative techniques, see [1-35] and the references therein. We would like to emphasize that due its applications, the convexity theory has been generalized and extended in different directions using novel and fascinating ideas. A significant generalization of convex functions is that of invex function introduced by Hanson [10]. This class of convex functions inspired many researchers to study a wide class of problems which arise in optimization and engineering sciences in a unified framework. Ben-Israel and Mond [3] introduced the concept of invex set and peinvex functions. They have shown that the differentiable preinvex functions are invex functions, but the converse is not true. Noor [18] proved that the minimum of the differentiable preinvex function on the invex sets can be characterized by a class of variational inequalities, called the variational-like inequalities. For the applications, formulation, numerical methods and other aspects of variational-like inequalities, see [20]. Pitea and Postolache [32, 33, 34] introduced the concept of quasi invexity and applied it to the theoretical mechanics and nonlinear optimization. This shows that the preinvexity and its variant generalizations play an important and significant role in the developments of various fields of pure and applied sciences. Noor[17] proved that a function f is a preinvex function, if and only if, it satisfies the Hermite-Hadamad type integral inequality. This result can be viewed as an analogous to the Hermite-Hadamard type inequalities for convex functions. These results have been proved to be the starting point for the recent activities in the field of preinvexity and related areas.

We would like to point out that the concept of the convex set is related to the weighted arithmetic means. Related to arithmetic means, we have concept of harmonic mean. The harmonic means have applications in electrical circuit theory and computer science. One defines the harmonic convex functions using the concepts of weighted harmonic means. Anderson et. al. [1] and Iscan [11] have introduced harmonic convex functions, which can be viewed as an important generalization of the convex functions. For the applications and characterizations of the harmonic convex functions, see [1, 11, 27].

We note that harmonic convex functions and preinvex functions are quite different generalizations of the convex functions. It i natural to unify these different concepts. Combining these concepts, Noor et al [24, 25] introduced and investigated the harmonic preinvex functions, which can be viewed as a unifying one. It have shown that the harmonic preinvex functions include preinvex functions and harmonic convex functions as special cases. For the recent developments in this area, see [2, 24, 25, 26] and the references therein.

To be more precise, let us consider a bidimensional interval $\Delta = [a,b]\times[c,d] \subset \mathbb{R}^2$ with $a < b$ and $c < d$. Dragomir [8] established the Hermite-Hadamard type inequality for coordinated convex functions in a rectangle from the plane $\mathbb{R}^2$. Bakula [4] established the refinement of the Hermite-Hadamard type inequality for co-ordinated convex functions.

Theorem 1.1. [4]. *Let $f : \Delta = [a,b]\times[c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ be co-ordinated convex function on the rectangle Δ. Then*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}y\right]\\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\mathrm{d}y\mathrm{d}x\\ &\leq \frac{1}{8}\left[\frac{1}{b-a}\int_a^b\left(f(x,c)+f(x,d)+2f\left(x,\frac{c+d}{2}\right)\right)\mathrm{d}x\right.\\ &\quad\left.+\frac{1}{d-c}\int_c^d\left(f(a,y)+f(b,y)+2f\left(\frac{a+b}{2},y\right)\right)\mathrm{d}y\right]\\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{16}+\frac{1}{4}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &\quad+\frac{f(a,\frac{c+d}{2})+f(b,\frac{c+d}{2})+f(\frac{a+b}{2},d)+f(\frac{a+b}{2},c)}{8}. \end{aligned} \tag{1}$$

Chen [7] has obtained some new Hermite-Hadamard type inequalities for co-ordinated convex functions on the co-ordinates, which may be viewed as a refinement of the Hermite-Hadamard inequality obtained by Bakula [4]. Noor et. al.[21] introduced the concept of coordinated harmonic convex functions.

Definition 1.1. [21]. *Consider a rectangle $\Delta = [a,b]\times[c,d] \subset \mathbb{R}^2 \setminus \{0\}$ with $a < b$ and $c < d$. A function $f : \Delta = [a,b]\times[c,d] \to \mathbb{R}$ is said to be harmonic convex* *function on the rectangle* Δ*, if*

$$f\left(\frac{ab}{\lambda a+(1-\lambda)b},\frac{cd}{\lambda c+(1-\lambda)d}\right)\leq(1-\lambda)f(a,c)+\lambda(b,d),$$

for all (a,b)*,* $(c,d)\in\Delta$ *and* $\lambda\in[0,1]$*.*

The co-ordinated harmonic convex functions may be defined as:

Definition 1.2. [21]. *Consider a rectangle* $\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\setminus\{0\}$ *with* $a<b$ *and* $c<d$*. A function* $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ *is said to be co-ordinated harmonic convex function on the rectangle* Δ*, if*

$$\begin{aligned}f\left(\frac{ab}{\lambda a+(1-\lambda)b},\frac{cd}{tc+(1-t)d}\right) &\leq (1-\lambda)(1-t)f(a,c)+(1-\lambda)tf(a,d)\\ &\quad+\lambda(1-t)f(b,c)+\lambda t(b,d),\end{aligned}$$

for all (a,b)*,* $(c,d)\in\Delta$ *and* $\lambda,t\in[0,1]$*.*

Noor et. al. [28] have obtained the following refinement of the Hermite-Hadamard inequality for coordinated harmonic convex functions.

Theorem 1.2. *Let* $f:\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ *be co-ordinated harmonic convex function on the rectangle* Δ*. Then for any* $\lambda\in[0,1]$ *and* $t\in[0,1]$*, we have*

$$\begin{aligned}f\left(\frac{2ab}{a+b},\frac{2cd}{c+d}\right) &\leq \phi(\lambda,t)\\ &\leq \frac{(ab)(cd)}{(b-a)(d-c)}\int_a^b\int_c^d\frac{f(x,y)}{x^2y^2}\mathrm{d}x\mathrm{d}y\\ &\leq \psi(\lambda,t)\\ &\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}$$

where

$$\begin{aligned}\phi(\lambda,t) &= t\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(2-t)c+td}\right)\\ &\quad+\lambda(1-t)f\left(\frac{2ab}{(2-\lambda)a+\lambda b},\frac{2cd}{(1-t)c+(1+t)d}\right)\\ &\quad+t(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(1+\lambda)b},\frac{2cd}{(2-t)c+td}\right)\\ &\quad+(1-\lambda)(1-t)f\left(\frac{2ab}{(1-\lambda)a+(1+\lambda)b},\frac{2cd}{(1-t)c+(1+t)d}\right).\end{aligned}$$

and

$$\begin{aligned}\psi(\lambda,t) &= \frac{t\lambda}{4}f(b,d)+\frac{\lambda(1-t)}{4}f(b,c)+\frac{t(1-\lambda)}{4}f(a,d)+\frac{(1-t)(1-\lambda)}{4}f(a,c)\\ &\quad+\frac{1}{4}f\left(\frac{ab}{(1-\lambda)a+\lambda b},\frac{cd}{(1-t)c+td}\right)+\frac{\lambda}{4}f\left(b,\frac{cd}{(1-t)c+td}\right)\\ &\quad+\frac{1-\lambda}{4}f\left(a,\frac{cd}{(1-t)c+td}\right)+\frac{t}{4}f\left(\frac{ab}{(1-\lambda)a+\lambda b},d\right)\\ &\quad+\frac{1-t}{4}f\left(\frac{ab}{(1-\lambda)a+\lambda b},c\right).\end{aligned}$$

Matloka [16] introduced the class of (h_1, h_2)-preinvex functions on the co-ordinates and obtained some new Hermite-Hadamard inequalities. To be more precise, let $X_1 \times X_2$ be nonempty subsets of $\mathbb{R}^n$, let $\eta_1(.,.) : X_1 \times X_2 \to \mathbb{R}^n$ and $\eta_2(.,.) : X_1 \times X_2 \to \mathbb{R}^n$ be bifunctions.

Definition 1.3. [16]. *Let $(u, v) \in X_1 \times X_2$. We say $X_1 \times X_2$ is invex at (u, v) with respect to η_1 and η_2 if for each $(x, y) \in X_1 \times X_2$ and t_1, $t_2 \in [0, 1]$,*

$$(u + t_1\eta_1(x, u), v + t_2\eta_2(y, v)) \in X_1 \times X_2.$$

$X_1 \times X_2$ is said to be an invex set with respect to η_1 and η_2 if $X_1 \times X_2$ is invex at each $(u, v) \in X_1 \times X_2$.

We now introduce some new classes of coordinated harmonic preinvex functions, which is one of the main purpose of this paper.

Definition 1.4. *Consider a rectangle $\Delta = [a, a+\eta_1(b, a)]\times[c, c+\eta_2(d, c)] \subset \mathbb{R}^2\setminus\{0\}$ with $a < (a + \eta_1(b, a))$ and $c < (c + \eta_2(d, c))$. A function $f : \Delta = [a, a + \eta_1(b, a)] \times [c, c+\eta_2(d, c)] \to \mathbb{R}$ is said to be harmonic preinvex function on the rectangle Δ with respect to η_1 and η_2, if*

$$f\left(\frac{ab}{a + (1-\lambda)\eta_1(b, a)}, \frac{cd}{c + (1-\lambda)\eta_2(d, c)}\right) \leq (1-\lambda)f(a, c) + \lambda(a + \eta_1(b, a), c + \eta_2(d, c)),$$

for all $(a, a + \eta_1(b, a))$, $(c, c + \eta_2(d, c)) \in \Delta$ and $\lambda \in [0, 1]$.

Definition 1.5. *Consider a rectangle $\Delta = [a, a+\eta_1(b, a)]\times[c, c+\eta_2(d, c)] \subset \mathbb{R}^2\setminus\{0\}$ with $a < (a + \eta_1(b, a))$ and $c < (c + \eta_2(d, c))$. A function $f : \Delta = [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \to \mathbb{R}$ is said to be co-ordinated harmonic preinvex function on the rectangle Δ with respect to η_1 and η_2, if*

$$\begin{aligned} & f\left(\frac{ab}{a + (1-\lambda)\eta_1(b, a)}, \frac{cd}{c + (1-t)\eta_2(d, c)}\right) \\ \leq\ & (1-\lambda)(1-t)f(a, c) + (1-\lambda)tf(a, c + \eta_2(d, c)) \\ & +\lambda(1-t)f(a + \eta_1(b, a), c) + \lambda t(a + \eta_1(b, a), c + \eta_2(d, c)), \end{aligned}$$

for all $(a, a + \eta_1(b, a))$, $(c, c + \eta_2(d, c)) \in \Delta$ and $\lambda, t \in [0, 1]$.

The following simple fact plays an important role in the derivation of the main results of this paper.

Remark 1.1. *If $I \subset \Delta = [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)] \subset \mathbb{R}^2 \setminus \{0\}$ and consider the function $g : \left[\frac{1}{a+\eta_1(b,a)}, \frac{1}{a}\right] \times \left[\frac{1}{c+\eta_2(d,c)}, \frac{1}{c}\right] \to \mathbb{R}$ defined by $g(s_1, s_2) = f\left(\frac{1}{s_1}, \frac{1}{s_2}\right)$, then f is co-ordinated harmonic preinvex on $[a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)]$ with respect to η_1 and η_2, if and only if, g is co-ordinated convex in the usual sense on $\left[\frac{1}{a+\eta_1(b,a)}, \frac{1}{a}\right] \times \left[\frac{1}{c+\eta_2(d,c)}, \frac{1}{c}\right] \to \mathbb{R}$.*

2. Main results

In this section, we obtain our main result. For simplicity, we take $\bar{b} = a + \eta_1(b, a)$ and $\bar{d} = c + \eta_1(b, a)$. In order to prove our main result, we need the following representation theorem.

Theorem 2.1. *Let $g : \Delta = [a, \bar{b}] \times [c, \bar{d}] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a Lebesgue integrable function on the rectangle Δ and $\lambda, t \in [0,1]$ with respect to η_1 and η_2, then*

$$\begin{aligned}
& \int_0^1 \int_0^1 g((1-t_1)a + t_1\bar{b}, (1-t_2)c + t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \lambda t \int_0^1 \int_0^1 g((1-t_1)a + t_1[(1-\lambda)a + \lambda\bar{b}], (1-t_2)c + t_2[(1-t)c + t\bar{d}])\mathrm{d}t_1\mathrm{d}t_2 \\
+ \;& \lambda(1-t) \int_0^1 \int_0^1 g((1-t_1)a + t_1[(1-\lambda)a + \lambda\bar{b}], (1-t_2)[(1-t)c + t\bar{d}] + t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2 \\
+ \;& t(1-\lambda) \int_0^1 \int_0^1 g((1-t_1)[(1-\lambda)a + \lambda\bar{b}] + t_1\bar{b}, (1-t_2)c + t_2[(1-t)c + t\bar{d}])\mathrm{d}t_1\mathrm{d}t_2 \\
+ \;& (1-t)(1-\lambda) \int_0^1 \int_0^1 g((1-t_1)[(1-\lambda)a + \lambda\bar{b}] + t_1\bar{b}, (1-t_2)[(1-t)c + t\bar{d}] + t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2.
\end{aligned} \tag{2}$$

Proof. For $\lambda = 0, t = 0$ and $\lambda = 1, t = 1$, the inequality (2) is obvious.
If $\lambda = 0$ and $t \in [0,1]$, then

$$\begin{aligned}
& \int_0^1 \int_0^1 g((1-t_1)a + t_1\bar{b}, (1-t_2)c + t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& t \int_0^1 \int_0^1 g((1-t_1)a + t_1\bar{b}, (1-t_2)c + t_2[(1-t)c + t\bar{d}])\mathrm{d}t_1\mathrm{d}t_2 \\
+ \;& (1-t) \int_0^1 \int_0^1 g((1-t_1)a + t_1\bar{b}, (1-t_2)[(1-t)c + t\bar{d}] + t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2.
\end{aligned}$$

If $t = 0$ and $\lambda \in [0,1]$, then

$$\begin{aligned}
& \int_0^1 \int_0^1 g((1-t_1)a + t_1\bar{b}, (1-t_2)c + t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \lambda \int_0^1 \int_0^1 g((1-t_1)a + t_1[(1-\lambda)a + \lambda\bar{b}], (1-t_2)c + t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2 \\
+ \;& (1-\lambda) \int_0^1 \int_0^1 g((1-t_1)[(1-\lambda)a + \lambda\bar{b}] + t_1\bar{b}, (1-t_2)c + t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2.
\end{aligned}$$

Also for $\lambda \in (0,1)$ and $t \in (0,1)$, we observe that

$$\begin{aligned}
(i). \quad & \lambda t \int_0^1 \int_0^1 g((1-t_1)a + t_1[(1-\lambda)a + \lambda\bar{b}], (1-t_2)c + t_2[(1-t)c + t\bar{d}])\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \lambda t \int_0^1 \int_0^1 g(t_1\lambda\bar{b} + (1-\lambda t_1)a, t_2 td + (1-tt_2)c)\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \int_0^\lambda \int_0^t g(u\bar{b} + (1-u)a, v\bar{d} + (1-v)c)\mathrm{d}v\mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
(ii). \quad & \lambda(1-t) \int_0^1 \int_0^1 g((1-t_1)a + t_1[(1-\lambda)a + \lambda\bar{b}], (1-t_2)[(1-t)c + t\bar{d}] + t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \lambda(1-t) \int_0^1 \int_0^1 g(t_1\lambda\bar{b} + (1-\lambda t_1)a, ((1-t_2)t + t_2)\bar{d} + (1-t_2)(1-t)c)\mathrm{d}t_1\mathrm{d}t_2 \\
= \;& \int_0^\lambda \int_t^1 g(u\bar{b} + (1-u)a, v\bar{d} + (1-v)c)\mathrm{d}v\mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
(iii). \quad & t(1-\lambda)\int_0^1\int_0^1 g((1-t_1)[(1-\lambda)a+\lambda\bar{b}]+t_1\bar{b},(1-t_2)c+t_2[(1-t)c+t\bar{d}])\mathrm{d}t_1\mathrm{d}t_2\\
= \quad & t(1-\lambda)\int_0^1\int_0^1 g(((1-t_1)\lambda+t_1)\bar{b}+(1-t_1)(1-\lambda)a,t_2td+(1-tt_2)c)\mathrm{d}t_1\mathrm{d}t_2\\
= \quad & \int_\lambda^1\int_0^t g(u\bar{b}+(1-u)a,v\bar{d}+(1-v)c)\mathrm{d}v\mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
(iv). \quad & (1-t)(1-\lambda)\int_0^1\int_0^1 g((1-t_1)[(1-\lambda)a+\lambda\bar{b}]+t_1\bar{b},(1-t_2)[(1-t)c+t\bar{d}]+t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2\\
= \quad & (1-t)(1-\lambda)\int_0^1\int_0^1 g(((1-t_1)\lambda+t_1)\bar{b}+(1-t_1)(1-\lambda)a,((1-t_2)t+t_2)\bar{d}\\
& +(1-t_2)(1-t)c)\mathrm{d}t_1\mathrm{d}t_2\\
= \quad & \int_\lambda^1\int_t^1 g(u\bar{b}+(1-u)a,v\bar{d}+(1-v)c)\mathrm{d}v\mathrm{d}u
\end{aligned}$$

Combining the above results, we have

$$\begin{aligned}
& \int_0^1\int_0^1 g((1-t_1)a+t_1\bar{b},(1-t_2)c+t_2\bar{d})\mathrm{d}t_1\mathrm{d}t_2\\
= & \int_0^\lambda\int_0^t g(u\bar{b}+(1-u)a,v\bar{d}+(1-v)c)\mathrm{d}v\mathrm{d}u+\int_0^\lambda\int_t^1 g(u\bar{b}+(1-u)a,v\bar{d}+(1-v)c)\mathrm{d}v\mathrm{d}u\\
+ & \int_\lambda^1\int_0^t g(u\bar{b}+(1-u)a,v\bar{d}+(1-v)c)\mathrm{d}v\mathrm{d}u+\int_\lambda^1\int_t^1 g(u\bar{b}+(1-u)a,v\bar{d}+(1-v)c)\mathrm{d}v\mathrm{d}u\\
= & \int_0^1\int_0^1 g(ub+(1-u)a,v\bar{d}+(1-v)c)\mathrm{d}v\mathrm{d}u
\end{aligned}$$

and the identity (2) is proved. □

We now prove Hermite-Hadamard inequality for co-ordinated harmonic convex functions.

Theorem 2.2. *Let $f:\Delta=[a,\bar{b}]\times[c,\bar{d}]\subset\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ be co-ordinated harmonic convex function on the rectangle Δ with respect to η_1 and η_2. Then for any $\lambda\in[0,1]$ and $t\in[0,1]$, we have*

$$\begin{aligned}
f\left(\frac{2a\bar{b}}{a+\bar{b}},\frac{2c\bar{d}}{c+\bar{d}}\right) &\le \phi(\lambda,t)\\
&\le \frac{(a\bar{b})(c\bar{d})}{(\bar{b}-a)(\bar{d}-c)}\int_a^{\bar{b}}\int_c^{\bar{d}}\frac{f(x,y)}{x^2y^2}\mathrm{d}x\mathrm{d}y\\
&\le \psi(\lambda,t)\\
&\le \frac{f(a,c)+f(a,\bar{d})+f(\bar{b},c)+f(\bar{b},\bar{d})}{4}. \qquad (3)
\end{aligned}$$

where

$$\begin{aligned}\phi(\lambda,t) &= t\lambda f\left(\frac{2a\bar{b}}{(2-\lambda)a+\lambda\bar{b}},\frac{2c\bar{d}}{(2-t)c+t\bar{d}}\right)\\ &+\lambda(1-t)f\left(\frac{2a\bar{b}}{(2-\lambda)a+\lambda\bar{b}},\frac{2c\bar{d}}{(1-t)c+(1+t)\bar{d}}\right)\\ &+t(1-\lambda)f\left(\frac{2a\bar{b}}{(1-\lambda)a+(1+\lambda)\bar{b}},\frac{2c\bar{d}}{(2-t)c+t\bar{d}}\right)\\ &+(1-\lambda)(1-t)f\left(\frac{2a\bar{b}}{(1-\lambda)a+(1+\lambda)\bar{b}},\frac{2c\bar{d}}{(1-t)c+(1+t)\bar{d}}\right).\end{aligned}$$

and

$$\begin{aligned}\psi(\lambda,t) &= \frac{t\lambda}{4}f(\bar{b},\bar{d})+\frac{\lambda(1-t)}{4}f(\bar{b},c)+\frac{t(1-\lambda)}{4}f(a,\bar{d})+\frac{(1-t)(1-\lambda)}{4}f(a,c)\\ &+\frac{1}{4}f\left(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},\frac{c\bar{d}}{(1-t)c+td}\right)+\frac{\lambda}{4}f\left(\bar{b},\frac{c\bar{d}}{(1-t)c+td}\right)\\ &+\frac{1-\lambda}{4}f\left(a,\frac{c\bar{d}}{(1-t)c+td}\right)+\frac{t}{4}f\left(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},\bar{d}\right)\\ &+\frac{1-t}{4}f\left(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},c\right).\end{aligned}$$

Proof. Let g be co-ordinated preinvex function on $\Delta=\left[\frac{1}{\bar{b}},\frac{1}{a}\right]\times\left[\frac{1}{\bar{d}},\frac{1}{c}\right]$, defined by $g(s_1,s_2)=f\left(\frac{1}{s_1},\frac{1}{s_2}\right)$, $s_1,s_2\in\left[\frac{1}{\bar{b}},\frac{1}{a}\right]\times\left[\frac{1}{\bar{d}},\frac{1}{c}\right]$, then using (1) and for $\lambda\in[0,1]$ and $t\in[0,1]$, we have

(*i*).

$$\begin{aligned}& g\left(\frac{(2-\lambda)a+\lambda\bar{b}}{2a\bar{b}},\frac{(2-t)c+t\bar{d}}{2c\bar{d}}\right)\\ =\;& g\left(\frac{\frac{1}{\bar{b}}+(1-\lambda)\frac{1}{\bar{b}}+\lambda\frac{1}{a}}{2},\frac{\frac{1}{\bar{d}}+(1-t)\frac{1}{\bar{d}}+t\frac{1}{c}}{2}\right)\\ \leq\;& \int_0^1\int_0^1 g\left((1-t_1)\frac{1}{\bar{b}}+t_1\left((1-\lambda)\frac{1}{\bar{b}}+\lambda\frac{1}{a}\right),(1-t_2)\frac{1}{\bar{d}}+t_2\left((1-t)\frac{1}{\bar{d}}+t\frac{1}{c}\right)\right)\mathrm{d}t_1\mathrm{d}t_2\\ \leq\;& \frac{g\left(\frac{1}{\bar{b}},\frac{1}{\bar{d}}\right)+g\left(\frac{1}{\bar{b}},(1-t)\frac{1}{\bar{d}}+t\frac{1}{c}\right)+g\left((1-\lambda)\frac{1}{\bar{b}}+\lambda\frac{1}{a},\frac{1}{\bar{d}}\right)+g\left((1-\lambda)\frac{1}{\bar{b}}+\lambda\frac{1}{a},(1-t)\frac{1}{\bar{d}}+t\frac{1}{c}\right)}{4}\\ =\;& \frac{g\left(\frac{1}{\bar{b}},\frac{1}{\bar{d}}\right)+g\left(\frac{1}{\bar{b}},\frac{(1-t)c+td}{c\bar{d}}\right)+g\left(\frac{(1-\lambda)a+\lambda\bar{b}}{a\bar{b}},\frac{1}{\bar{d}}\right)+g\left(\frac{(1-\lambda)a+\lambda\bar{b}}{a\bar{b}},\frac{(1-t)c+td}{c\bar{d}}\right)}{4}\end{aligned}\tag{4}$$

(*ii*).

$$\begin{aligned}& g\left(\frac{(2-\lambda)a+\lambda\bar{b}}{2a\bar{b}},\frac{(1-t)c+(1+t)\bar{d}}{2c\bar{d}}\right)\\ =\;& g\left(\frac{\frac{1}{\bar{b}}+(1-\lambda)\frac{1}{\bar{b}}+\lambda\frac{1}{a}}{2},\frac{(1-t)\frac{1}{\bar{d}}+t\frac{1}{c}+\frac{1}{c}}{2}\right)\\ \leq\;& \int_0^1\int_0^1 g\left((1-t_1)\frac{1}{\bar{b}}+t_1\left((1-\lambda)\frac{1}{\bar{b}}+\lambda\frac{1}{a}\right),(1-t_2)\left((1-t)\frac{1}{\bar{d}}+t\frac{1}{c}\right)+t_2\frac{1}{c}\right)\mathrm{d}t_1\mathrm{d}t_2\\ \leq\;& \frac{g\left(\frac{1}{\bar{b}},(1-t)\frac{1}{\bar{d}}+t\frac{1}{c}\right)+g\left(\frac{1}{\bar{b}},\frac{1}{c}\right)+g\left((1-\lambda)\frac{1}{\bar{b}}+\lambda\frac{1}{a},(1-t)\frac{1}{\bar{d}}+t\frac{1}{c}\right)+g\left((1-\lambda)\frac{1}{\bar{b}}+\lambda\frac{1}{a},\frac{1}{c}\right)}{4}\\ =\;& \frac{g\left(\frac{1}{\bar{b}},\frac{(1-t)c++td}{c\bar{d}}\right)+g\left(\frac{1}{\bar{b}},\frac{1}{c}\right)+g\left(\frac{(1-\lambda)a+\lambda\bar{b}}{a\bar{b}},\frac{(1-t)c++td}{c\bar{d}}\right)+g\left(\frac{(1-\lambda)a+\lambda\bar{b}}{a\bar{b}},\frac{1}{c}\right)}{4}\end{aligned}\tag{5}$$

$$
\begin{aligned}
(iii).\quad & g\Big(\frac{(1-\lambda)a+(\lambda+1)\bar{b}}{2a\bar{b}},\frac{(2-t)c+t\bar{d}}{2c\bar{d}}\Big)\\
= \ & g\Big(\frac{(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}+\frac{1}{a}}{2},\frac{\frac{1}{d}+(1-t)\frac{1}{d}+t\frac{1}{c}}{2}\Big)\\
\leq \ & \int_0^1\int_0^1 g\Big((1-t_1)\Big((1-\lambda)\frac{1}{\bar{b}}+\lambda\frac{1}{a}\Big)+t_1\frac{1}{a},(1-t_2)\frac{1}{\bar{d}}+t_2\Big((1-t)\frac{1}{\bar{d}}+t\frac{1}{c}\Big)\Big)\mathrm{d}t_1\mathrm{d}t_2\\
\leq \ & \frac{g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},\frac{1}{d}\big)+g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},(1-t)\frac{1}{d}+t\frac{1}{c}\big)+g\big(\frac{1}{a},\frac{1}{d}\big)+g\big(\frac{1}{a},(1-t)\frac{1}{d}+t\frac{1}{c}\big)}{4}\\
=(6)\ & \frac{g\big(\frac{(1-\lambda)a+\lambda\bar{b}}{a\bar{b}},\frac{1}{d}\big)+g\big(\frac{(1-\lambda)a+\lambda\bar{b}}{a\bar{b}},\frac{(1-t)c+td}{c\bar{d}}\big)+g\big(\frac{1}{a},\frac{1}{d}\big)+g\big(\frac{1}{a},\frac{(1-t)c+td}{c\bar{d}}\big)}{4}
\end{aligned}
$$

$$
\begin{aligned}
(iv).\quad & g\Big(\frac{(1-\lambda)a+(\lambda+1)\bar{b}}{2a\bar{b}},\frac{(1-t)c+(1+t)\bar{d}}{2c\bar{d}}\Big)\\
= \ & g\Big(\frac{(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}+\frac{1}{a}}{2},\frac{(1-t)\frac{1}{d}+t\frac{1}{c}+\frac{1}{c}}{2}\Big)\\
\leq \ & \int_0^1\int_0^1 g\Big((1-t_1)\Big((1-\lambda)\frac{1}{\bar{b}}+\lambda\frac{1}{a}\Big)+t_1\frac{1}{a},(1-t_2)\Big((1-t)\frac{1}{\bar{d}}+t\frac{1}{c}\Big)+t_2\frac{1}{c}\Big)\mathrm{d}t_1\mathrm{d}t_2\\
\leq \ & \frac{g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},(1-t)\frac{1}{d}+t\frac{1}{c}\big)+g\big((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a},\frac{1}{c}\big)+g\big(\frac{1}{a},(1-t)\frac{1}{d}+t\frac{1}{c}\big)+g\big(\frac{1}{a},\frac{1}{c}\big)}{4}\\
=(7)\ & \frac{g\big(\frac{(1-\lambda)a+\lambda\bar{b}}{a\bar{b}},\frac{(1-t)c+td}{cd}\big)+g\big(\frac{(1-\lambda)a+\lambda\bar{b}}{a\bar{b}},\frac{1}{c}\big)+g\big(\frac{1}{a},\frac{(1-t)c+td}{cd}\big)+g\big(\frac{1}{a},\frac{1}{c}\big)}{4}
\end{aligned}
$$

Multiply (4) by λ, (5)by $\lambda(1-t)$, (6) by $t(1-\lambda)$ and (7) by $(1-\lambda)(1-t)$ and adding the resultants, we have

$$
\begin{aligned}
& t\lambda f\Big(\frac{2a\bar{b}}{(2-\lambda)a+\lambda\bar{b}},\frac{2c\bar{d}}{(2-t)c+t\bar{d}}\Big)+\lambda(1-t)f\Big(\frac{2a\bar{b}}{(2-\lambda)a+\lambda\bar{b}},\frac{2c\bar{d}}{(1-t)c+(1+t)\bar{d}}\Big)\\
& +t(1-\lambda)f\Big(\frac{2a\bar{b}}{(1-\lambda)a+(1+\lambda)\bar{b}},\frac{2c\bar{d}}{(2-t)c+t\bar{d}}\Big)\\
& +(1-\lambda)(1-t)f\Big(\frac{2a\bar{b}}{(1-\lambda)a+(1+\lambda)\bar{b}},\frac{2c\bar{d}}{(1-t)c+(1+t)\bar{d}}\Big)\\
= \ & \phi(\lambda,t)\\
\leq \ & \int_0^1\int_0^1 f\Big(\frac{a\bar{b}}{(1-t_1)a+t_1\bar{b}},\frac{c\bar{d}}{(1-t_2)c+t_2\bar{d}}\Big)\mathrm{d}t_1\mathrm{d}t_2\\
= \ & \frac{(a\bar{b})(c\bar{d})}{(\bar{b}-a)(\bar{d}-c)}\int_a^{\bar{b}}\int_c^{\bar{d}}\frac{f(x,y)}{x^2y^2}\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{\lambda t}{4}\bigg[f(\bar{b},\bar{d})+f\bigg(\bar{b},\frac{c\bar{d}}{(1-t)c+td}\bigg)+f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},\bar{d}\bigg)\\
&+f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},\frac{c\bar{d}}{(1-t)c+td}\bigg)\bigg]+\frac{\lambda(1-t)}{4}\bigg[f\bigg(\bar{b},\frac{c\bar{d}}{(1-t)c++td}\bigg)+f(\bar{b},c)\\
&+f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},\frac{c\bar{d}}{(1-t)c++td}\bigg)+f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},c\bigg)\bigg]\\
&+\frac{t(1-\lambda)}{4}\bigg[f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},\bar{d}\bigg)+f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},\frac{c\bar{d}}{(1-t)c+td}\bigg)\\
&+f(a,\bar{d})+f\bigg(a,\frac{c\bar{d}}{(1-t)c+td}\bigg)\bigg]+\frac{(1-\lambda)(1-t)}{4}\bigg[f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},\frac{c\bar{d}}{(1-t)c+td}\bigg)\\
&+f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},c\bigg)+f\bigg(a,\frac{c\bar{d}}{(1-t)c+td}\bigg)+f(a,c)\bigg]\\
=\ & \frac{t\lambda}{4}f(\bar{b},\bar{d})+\frac{\lambda(1-t)}{4}f(\bar{b},c)+\frac{t(1-\lambda)}{4}f(a,\bar{d})+\frac{(1-t)(1-\lambda)}{4}f(a,c)\\
&+\frac{1}{4}f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},\frac{c\bar{d}}{(1-t)c+td}\bigg)+\frac{\lambda}{4}f\bigg(\bar{b},\frac{c\bar{d}}{(1-t)c+td}\bigg)+\frac{1-\lambda}{4}f\bigg(a,\frac{c\bar{d}}{(1-t)c+td}\bigg)\\
&+\frac{t}{4}f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},\bar{d}\bigg)+\frac{1-t}{4}f\bigg(\frac{a\bar{b}}{(1-\lambda)a+\lambda\bar{b}},c\bigg)\\
=\ & (8)\psi(\lambda,t).
\end{aligned}
$$

Also

$$
\begin{aligned}
\phi(\lambda,t) =\ & t\lambda f\bigg(\frac{2a\bar{b}}{(2-\lambda)a+\lambda\bar{b}},\frac{2c\bar{d}}{(2-t)c+t\bar{d}}\bigg)\\
&+\lambda(1-t)f\bigg(\frac{2a\bar{b}}{(2-\lambda)a+\lambda\bar{b}},\frac{2c\bar{d}}{(1-t)c+(1+t)\bar{d}}\bigg)\\
&+t(1-\lambda)f\bigg(\frac{2a\bar{b}}{(1-\lambda)a+(1+\lambda)\bar{b}},\frac{2c\bar{d}}{(2-t)c+t\bar{d}}\bigg)\\
&+(1-\lambda)(1-t)f\bigg(\frac{2a\bar{b}}{(1-\lambda)a+(1+\lambda)\bar{b}},\frac{2c\bar{d}}{(1-t)c+(1+t)\bar{d}}\bigg)\\
=\ & t\lambda g\bigg(\frac{(2-\lambda)a+\lambda\bar{b}}{2a\bar{b}},\frac{(2-t)c+t\bar{d}}{2c\bar{d}}\bigg)\\
&+\lambda(1-t)g\bigg(\frac{(2-\lambda)a+\lambda\bar{b}}{2a\bar{b}},\frac{(1-t)c+(1+t)\bar{d}}{2c\bar{d}}\bigg)\\
&+t(1-\lambda)g\bigg(\frac{(1-\lambda)a+(1+\lambda)\bar{b}}{2a\bar{b}},\frac{(2-t)c+t\bar{d}}{2c\bar{d}}\bigg)\\
&+(1-\lambda)(1-t)g\bigg(\frac{(1-\lambda)a+(1+\lambda)\bar{b}}{2a\bar{b}},\frac{(1-t)c+(1+t)\bar{d}}{2c\bar{d}}\bigg)\\
\geq\ & g\bigg((1-\lambda)\bigg(\frac{(1-\lambda)a+(1+\lambda)\bar{b}}{2a\bar{b}}\bigg)+\lambda\bigg(\frac{(2-\lambda)a+\lambda\bar{b}}{2a\bar{b}}\bigg),\\
&(1-t)\bigg(\frac{(1-t)c+(1+t)\bar{d}}{2c\bar{d}}\bigg)+t\bigg(\frac{(2-t)c+td}{2c\bar{d}}\bigg)\bigg)
\end{aligned}
$$

$$
(9)\qquad = g\bigg(\frac{a+\bar{b}}{2a\bar{b}},\frac{c+\bar{d}}{2c\bar{d}}\bigg)=f\bigg(\frac{2a\bar{b}}{a+\bar{b}},\frac{2c\bar{d}}{c+\bar{d}}\bigg)
$$

and

$$\begin{aligned}\psi(\lambda,t) &\leq \frac{t\lambda}{4}f(\bar{b},\bar{d})+\frac{\lambda(1-t)}{4}f(\bar{b},c)+\frac{t(1-\lambda)}{4}f(a,\bar{d})+\frac{(1-t)(1-\lambda)}{4}f(a,c)\\ &+\frac{(1-\lambda)(1-t)}{4}f(\bar{b},\bar{d})+\frac{\lambda t}{4}f(a,c)+\frac{\lambda(1-t)}{4}f(a,\bar{d})+\frac{t(1-\lambda)}{4}f(\bar{b},c)\\ &+\frac{\lambda t}{4}f(\bar{b},c)+\frac{\lambda(1-t)}{4}f(\bar{b},\bar{d})+\frac{\lambda t}{4}f(a,\bar{d})+\frac{t(1-\lambda)}{4}f(\bar{b},\bar{d})\\ &+\frac{(1-t)(1-\lambda)}{4}f(\bar{b},c)+\frac{(1-t)\lambda}{4}f(a,c)+\frac{(1-\lambda)(1-t)}{4}f(a,\bar{d})\\ &+\frac{(1-\lambda)t}{4}f(a,c)\\ &= \frac{f(a,c)+f(a,\bar{d})+f(\bar{b},c)+f(\bar{b},\bar{d})}{4}. \qquad (10)\end{aligned}$$

From (8)-(10), we obtained the desired inequality (3). □

We would like to mention that for $\lambda=\frac{1}{2}$ and $t=\frac{1}{2}$, Theorem 2.2 reduces to the following new result, which can be viewed as an important and significant extension of Bakula's result [4] for coordinated harmonic preinvex functions.

Corollary 2.3. *Let $f:\Delta=[a,\bar{b}]\times[c,\bar{d}]\subset\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ is co-ordinated harmonic preinvex convex function on the rectangle Δ with respect to η_1 and η_2. Then*

$$\begin{aligned}f\left(\frac{2a\bar{b}}{a+\bar{b}},\frac{2c\bar{d}}{c+\bar{d}}\right) &\leq \frac{1}{4}\left[f\left(\frac{4a\bar{b}}{3a+\bar{b}},\frac{4c\bar{d}}{3c+\bar{d}}\right)+f\left(\frac{4a\bar{b}}{3a+\bar{b}},\frac{4c\bar{d}}{c+3\bar{d}}\right)\right.\\ &+ \left. f\left(\frac{4a\bar{b}}{a+3\bar{b}},\frac{4c\bar{d}}{3c+\bar{d}}\right)+f\left(\frac{4a\bar{b}}{a+3\bar{b}},\frac{4c\bar{d}}{c+3\bar{d}}\right)\right]\\ &\leq \frac{(a\bar{b})(c\bar{d})}{(\bar{b}-a)(\bar{d}-c)}\int_a^{\bar{b}}\int_c^{\bar{d}}\frac{f(x,y)}{x^2y^2}\mathrm{d}x\mathrm{d}y\\ &\leq \frac{f(\bar{b},\bar{d})+f(\bar{b},c)+f(a,\bar{d})+f(a,c)}{16}+\frac{1}{4}f\left(\frac{2a\bar{b}}{a+\bar{b}},\frac{2c\bar{d}}{c+\bar{d}}\right)\\ &+ \frac{f(\bar{b},\frac{2c\bar{d}}{c+\bar{d}})+f(a,\frac{2c\bar{d}}{c+\bar{d}})+f(\frac{2a\bar{b}}{a+\bar{b}},\bar{d})+f(\frac{2a\bar{b}}{a+\bar{b}},c)}{8}\\ &\leq \frac{f(a,c)+f(a,\bar{d})+f(\bar{b},c)+f(\bar{b},\bar{d})}{4}. \qquad (11)\end{aligned}$$

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